

Evolutional entanglement production

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Abstract

Evolutional entanglement production is defined as the amount of entanglement produced by the evolution operator. This quantity is analyzed for systems whose Hamiltonians are characterized by spin operators. The evolutional entanglement production at the initial stage grows quadratically in time. For longer times, it oscillates, being quasiperiodic or periodic depending on the system parameters.

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1 Introduction

The notion of entanglement is at the center of several interrelated problems, such as quantum information processing, quantum computing, quantum measurements, and quantum decision theory [1–6]. A closely related notion is *entanglement production* that characterizes the amount of entanglement produced by quantum operations [7–16].

The difference between the notions of *state entanglement* and *entanglement production* is as follows. The state entanglement characterizes the structure of a given state. For example, whether the system state can be represented as a product of partial states or not [1–5]. While entanglement production, induced by a quantum operation, shows whether this operation transforms a disentangled state to an entangled one or not. To be more precise, let us consider a system associated with a Hilbert space \mathcal{H} and let the system be decomposable onto parts associated with Hilbert spaces \mathcal{H}_i . Let the system be described by a disentangled state of the product form

$$\psi_{dis} = \bigotimes_i \psi_i ,$$

in which $\psi_i \in \mathcal{H}_i$. But assume that we need to transform the disentangled state into an entangled one. The necessity of transforming a disentangled state into an entangled state can be dictated by the desire of using the entangled state for some applications in quantum information processing or quantum computing, where the use of entangled states is known to be essentially more efficient than the use of disentangled states [1–5]. The required transformation can be realized by an operation described by an operator, say \hat{A} , such that its action on the given disentangled state yields an entangled state

$$\psi_{ent} = \hat{A}\psi_{dis}$$

that cannot be represented in the product form of partial states.

For a single given disentangled state, it is possible to find an appropriate operator. However, the typical question, accompanying the process of such transformations, is: How would it be possible to find an operation that would be the most efficient for entangling, not just one given state, but the states from the whole class of disentangled states of the considered Hilbert space? To answer this question, it is necessary to have a characteristic quantifying the ability of different operators to produce entangled states. Such a characteristic has been given by the measure of entanglement production of quantum operations [9, 10]. In these papers, the use of the introduced measure was illustrated by numerous cases of pure as well as mixed states. It was also shown that entanglement production by reduced density operators can be employed for characterizing phase transitions in statistical systems, so that thermodynamic phase transitions are usually accompanied by *entanglement transitions*. For instance, in Bose-Einstein condensation, entanglement production by density operators decreases, which also happens in paramagnetic to ferromagnetic transition, contrary to the increase of the related measure in the transition from normal metal to superconductor. In Ref. [11], it was shown that entanglement produced by atomic correlations through the common radiation field experiences sharp peaks in the regime of electromagnetic superradiance. In Refs. [12, 17], it was demonstrated that entanglement can be produced in a Bose-condensed system by an external alternating field creating multiple coherent topological modes. The same can be done by shaking an optical lattice filled with Bose-Einstein condensate [18]. The consequences of entanglement production can be noticed in time-of-flight experiments [12].

In Refs. [6, 19], it was studied how the process of quantum measurements can produce entanglement in a multi-mode quantum system.

Instead of producing entanglement by some operations, it is possible to let the given disentangled state naturally evolve in time until it becomes entangled. Such a process is described by the evolution operator $\hat{U}(t)$, with the corresponding evolution generator playing the role of the system Hamiltonian. In this case, a system, starting from an initial non-entangled state can become entangled in the process of its natural evolution with the given Hamiltonian, so that

$$\psi_{ent}(t) = \hat{U}(t)\psi_{dis}(0) .$$

This kind of time-dependent entanglement can be described by a measurement procedure accomplished in a sequence of times with calculating, e.g., concurrence at these different time moments [20]. This method gives the sequence of values characterizing the state entanglement at different times. The efficiency of entanglement production for given initial and final states can be associated with the entanglement probability

$$p_{ent}(t) \equiv |(\psi_{ent}(t) | \psi_{dis}(0))|^2 = |(\psi_{dis}(0) | \hat{U}(t)\psi_{dis}(0))|^2 ,$$

introduced by analogy with the transition probability. The quantum transition probabilities of the type

$$p(\psi_1 \rightarrow \psi_2) \equiv |(\psi_2 | \psi_1)|^2$$

are widely used in numerous applications characterizing different quantum transitions, return probability, and quantum many-body localization [21–25]. However, such probabilities are defined for the given pair of an initial and final states, being strongly dependent on them. Again, aiming at quantifying the entangling properties of the evolution operator, not just for a given pair of states, but for a whole class of states from the considered Hilbert space, we can employ the measure of entanglement production introduced in Refs. [9, 10].

It is the aim of the present paper to define entanglement production caused by the evolution operator and to study its temporal behavior for some concrete examples. For this illustration, we choose the systems that are characterized by Hamiltonians expressed through spin operators. Such a type of Hamiltonians is generic for many systems describing finite-level or finite-state physical objects. Many finite quantum systems can be approximated by finite-level models, when only several low-lying energy levels are involved in the studied physical processes [26]. The entanglement production by the evolution operator has not been considered in the previous papers.

2 Measure of entanglement production

Let us consider a system characterized by the Hilbert space

$$\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i , \tag{1}$$

where each of the spaces \mathcal{H}_i is a closed linear envelope of an orthonormal basis of microstates,

$$\mathcal{H}_i = \text{span}\{|n_i\rangle\} .$$

Then the basis in space (1) is formed by the states

$$|n_1 n_2 \dots n_N\rangle \equiv \bigotimes_{i=1}^N |n_i\rangle , \quad (2)$$

so that

$$\mathcal{H} = \text{span} \left\{ \bigotimes_{i=1}^N |n_i\rangle \right\} . \quad (3)$$

Among the states of space (3), it is possible to separate two types of qualitatively different states, disentangled and entangled. The set of disentangled states

$$\mathcal{D} \equiv \left\{ \varphi = \bigotimes_{i=1}^N \varphi_i \right\} \subset \mathcal{H} \quad (4)$$

is formed by the states that are represented as factors of the partial states

$$\varphi_i = \sum_{n_i} c_{n_i} |n_i\rangle \in \mathcal{H}_i .$$

The states that cannot be represented as such factor states are called entangled.

Let us be interested in the action of an operator \hat{A} , with a nonzero trace, acting on space (3). Generally, its action on a state $\varphi \in \mathcal{H}$ can produce an entangled state, even when the state φ is disentangled. The measure of this entanglement production can be quantified in the following way [9, 10]. For a given operator \hat{A} , we define its non-entangling counterpart

$$\hat{A}^\otimes \equiv \frac{\bigotimes_{i=1}^N \hat{A}_i}{(\text{Tr}_{\mathcal{H}} \hat{A})^{N-1}} , \quad (5)$$

in which a partial factor operator

$$\hat{A}_i \equiv \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{A} \quad (6)$$

is obtained by taking the trace of \hat{A} over all spaces \mathcal{H}_j , composing \mathcal{H} , except the single space \mathcal{H}_i . The so-defined non-entangling counterpart (5) enjoys the same normalization as \hat{A} , so that

$$\text{Tr}_{\mathcal{H}} \hat{A}^\otimes = \text{Tr}_{\mathcal{H}} \hat{A} . \quad (7)$$

For what follows, we need the definition of an operator norm. We opt for the Hilbert-Schmidt norm that for an operator \hat{A} reads as

$$\|\hat{A}\|_{\mathcal{H}} \equiv \sqrt{\text{Tr}_{\mathcal{H}} (\hat{A}^\dagger \hat{A})} \equiv \|\hat{A}\| .$$

Respectively, for an operator \hat{A}_i on the space \mathcal{H}_i , the norm is

$$\|\hat{A}_i\|_{\mathcal{H}_i} \equiv \sqrt{\text{Tr}_{\mathcal{H}_i} (\hat{A}_i^\dagger \hat{A}_i)} \equiv \|\hat{A}_i\| .$$

This norm, also termed the Frobenius norm or Schur norm, is analogous to the Euclidean norm for vectors. It is a particular case ($p = 2$) of the Schatten p -norm, and, as all Schatten

norms, it is invariant under unitary transformations [27, 28], thus, does not depend on the chosen basis.

The measure of entanglement production for an operator \hat{A} is defined [9, 10] as

$$\varepsilon(\hat{A}) \equiv \log \frac{\|\hat{A}\|}{\|\hat{A}^{\otimes}\|} , \quad (8)$$

where the logarithm can be taken with respect to any convenient base. This quantity satisfies all properties required for being considered as a measure [9, 10]. In particular, when the operator \hat{A} is not entangling, then $\varepsilon(\hat{A}) = 0$.

For the norm of the non-entangling operator (5), we have

$$\|\hat{A}^{\otimes}\|^2 = \frac{\prod_{i=1}^N \|\hat{A}_i\|^2}{|\text{Tr}_{\mathcal{H}\hat{A}}|^2(N-1)} . \quad (9)$$

Therefore the measure of entanglement production (8) can be represented in the form

$$\varepsilon(\hat{A}) = \log \left\{ \|\hat{A}\| \cdot |\text{Tr}_{\mathcal{H}\hat{A}}|^{N-1} \prod_{i=1}^N \|\hat{A}_i\|^{-1} \right\} . \quad (10)$$

It is important that the defined measure (8) or (10) is very general, being introduced for arbitrary operators with non-zero trace and for arbitrary systems, whether pure or mixed, bipartite or multipartite. Many examples of its application to concrete physical systems can be found in Refs. [9–12, 17–19]. More concretely, the results of the previous papers are described in the Introduction. We emphasize that the entanglement production by the evolution operator has not been considered earlier.

3 Entangling and non-entangling operators

Definition. *An operator \hat{A} on a Hilbert space \mathcal{H} is called entangling if, acting on some disentangled states from this Hilbert space \mathcal{H} , it produces entangled states, as a result of which its measure of entanglement production is nonzero. But when the action of the operator \hat{A} on any disentangled state from \mathcal{H} produces another disentangled state, so that the operator entanglement production measure is zero, such an operator is termed non-entangling.*

In order to clearly illustrate how an operator can produce an entangled state from a disentangled state, let us consider a bipartite system characterized by the Hilbert space

$$\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2 = \text{span} \left\{ |n\alpha\rangle = |n\rangle \otimes |\alpha\rangle \right\} , \quad (11)$$

composed of two subsystems described by the Hilbert spaces

$$\mathcal{H}_1 = \text{span}\{|n\rangle\} , \quad \mathcal{H}_2 = \text{span}\{|\alpha\rangle\} . \quad (12)$$

An operator \hat{A} , acting on space (11), can be represented as a resolution

$$\hat{A} = \sum_{mn} \sum_{\alpha\beta} A_{mn}^{\alpha\beta} |m\alpha\rangle \langle n\beta| , \quad (13)$$

where

$$A_{mn}^{\alpha\beta} \equiv \langle m\alpha | \hat{A} | n\beta \rangle .$$

We assume that the operator possesses a nontrivial trace

$$\text{Tr}_{\mathcal{H}} \hat{A} = \sum_{n\alpha} A_{nn}^{\alpha\alpha} \neq 0 .$$

The disentangled set consists of disentangled states,

$$\mathcal{D} \equiv \left\{ \varphi_{dis} = \varphi_1 \otimes \varphi_2 \right\} . \quad (14)$$

In view of the expansions

$$\varphi_1 = \sum_n a_n |n\rangle \in \mathcal{H}_1 , \quad \varphi_2 = \sum_{\alpha} b_{\alpha} |\alpha\rangle \in \mathcal{H}_2 ,$$

the disentangled state can be written as

$$\varphi_{dis} \equiv \varphi_1 \otimes \varphi_2 = \sum_{n\alpha} a_n b_{\alpha} |n\alpha\rangle . \quad (15)$$

The action of operator (13) on the disentangled state (15) gives

$$\hat{A} \varphi_{dis} = \sum_{mn} \sum_{\alpha\beta} A_{mn}^{\alpha\beta} a_n b_{\alpha} |m\alpha\rangle . \quad (16)$$

The resulting state (16) is entangled if

$$A_{mn}^{\alpha\beta} \neq \delta_{mn} \delta_{\alpha\beta} A_n B_{\alpha} .$$

In other words, the operator is entangling, provided it cannot be represented in the form

$$\hat{A} \neq \left(\sum_n A_n |n\rangle \langle n| \right) \otimes \left(\sum_{\alpha} B_{\alpha} |\alpha\rangle \langle \alpha| \right) .$$

For the partial factor operators, we have

$$\hat{A}_1 \equiv \text{Tr}_{\mathcal{H}_2} \hat{A} = \sum_{mn} \sum_{\alpha} A_{mn}^{\alpha\alpha} |m\rangle \langle n| , \quad \hat{A}_2 \equiv \text{Tr}_{\mathcal{H}_1} \hat{A} = \sum_n \sum_{\alpha\beta} A_{nn}^{\alpha\beta} |\alpha\rangle \langle \beta| .$$

Then the non-entangling counterpart (5) becomes

$$\hat{A}^{\otimes} = \frac{\hat{A}_1 \otimes \hat{A}_2}{\text{Tr}_{\mathcal{H}} \hat{A}} . \quad (17)$$

This yields the measure of entanglement production (10), with the norms

$$||\hat{A}_1||^2 = \sum_{mn} \sum_{\alpha\beta} (A_{mn}^{\alpha\alpha})^* A_{mn}^{\beta\beta} , \quad ||\hat{A}_2||^2 = \sum_{mn} \sum_{\alpha\beta} A_{mm}^{\alpha\beta} (A_{nn}^{\alpha\beta})^* ,$$

$$||\hat{A}||^2 = \sum_{mn} \sum_{\alpha\beta} |A_{mn}^{\alpha\beta}|^2 .$$

To show by a simple example how an operator can entangle an initially disentangled state, let us take the operator

$$\hat{A} = C \sum_{mn} |mm\rangle\langle nn| , \quad (18)$$

where C is a constant. Since

$$A_{mn}^{\alpha\beta} = C \delta_{m\alpha} \delta_{n\beta} ,$$

we find that the action of this operator on a disentangled state (15) results in the state

$$\hat{A}\varphi_{dis} = C \left(\sum_n a_n b_n \right) \sum_m |mm\rangle . \quad (19)$$

This is what is called a multimode state, which is a maximally entangled state. In the case of only two modes, it represents the well known Bell state.

To calculate the entanglement production measure in the case of many modes, we denote their number by M , given by the condition

$$M \equiv \dim \mathcal{H}_1 = \dim \mathcal{H}_2 .$$

Then we have

$$||\hat{A}||^2 = M^2 |C|^2 , \quad \text{Tr}_{\mathcal{H}} \hat{A} = MC .$$

For the partial factor operators

$$\hat{A}_i \equiv \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{A} = C \sum_n |n\rangle\langle n| ,$$

we get the norms squared

$$||\hat{A}_i||^2 = |C|^2 .$$

Therefore the norm squared of the non-entangling counterpart (17) is

$$||\hat{A}^\otimes||^2 = \frac{|C|^2}{M^2} .$$

In this way, we come to the measure of entanglement production,

$$\varepsilon(\hat{A}) = 2 \log M , \quad (20)$$

caused by operator (18).

4 Entangling by evolution operators

Evolution operators can produce entanglement in the process of natural system evolution. Suppose, at the initial time $t = 0$ the system is prepared in a disentangled state $\psi(0)$. In the process of its evolution, it passes to a state $\psi(t)$ that can be entangled by the action of the evolution operator, since

$$\psi(t) = \hat{U}(t)\psi(0) , \quad \hat{U}(t) = e^{-iHt} , \quad (21)$$

where H is the system Hamiltonian assumed to be independent of time. Then the produced entanglement can be quantified by the measure of entanglement production (8) or (10), with the evolution operator in the place of \hat{A} .

For concreteness, let us take the system Hamiltonian in the form

$$H = H_1 \otimes \hat{1}_2 + \hat{1}_1 \otimes H_2 + H_{int} , \quad (22)$$

characterizing two subsystems with the Hamiltonians H_1 and H_2 , defined on the Hilbert spaces \mathcal{H}_1 and \mathcal{H}_2 , respectively, so that

$$\begin{aligned} H_1|n\rangle &= E_n|n\rangle , & \mathcal{H}_1 &= \text{span}\{|n\rangle\} , \\ H_2|\alpha\rangle &= E_\alpha|\alpha\rangle , & \mathcal{H}_2 &= \text{span}\{|\alpha\rangle\} , \end{aligned} \quad (23)$$

and interacting by means of an interaction Hamiltonian H_{int} . The notation $\hat{1}_i$ implies a unity operator on the corresponding space \mathcal{H}_i . The system Hamiltonian (22) acts on the Hilbert space (11). Let the initial state be disentangled, being represented as

$$\psi(0) = \varphi_1 \otimes \varphi_2 \in \mathcal{D} . \quad (24)$$

The partial evolution operators are

$$\hat{U}_1(t) \equiv \text{Tr}_{\mathcal{H}_2} \hat{U}(t) , \quad \hat{U}_2(t) \equiv \text{Tr}_{\mathcal{H}_1} \hat{U}(t) . \quad (25)$$

The non-entangling evolution counterpart is of form (17), being

$$\hat{U}^\otimes(t) = \frac{\hat{U}_1(t) \otimes \hat{U}_2(t)}{\text{Tr}_{\mathcal{H}} \hat{U}(t)} . \quad (26)$$

For the measure of entanglement production, we get

$$\varepsilon(\hat{U}(t)) = \log \frac{||\hat{U}(t)||}{||\hat{U}^\otimes(t)||} \equiv \varepsilon(t) . \quad (27)$$

The evolution-operator norm is

$$||\hat{U}(t)||^2 = M_1 M_2 , \quad (28)$$

with the space dimensionalities denoted as

$$M_i \equiv \dim \mathcal{H}_i \quad (i = 1, 2) . \quad (29)$$

Thus, measure (27) becomes

$$\varepsilon(t) = \frac{1}{2} \log \frac{M_1 M_2}{||\hat{U}^\otimes(t)||^2} . \quad (30)$$

At the initial moment of time, before the evolution has started, the measure of entanglement production has to be zero. To show this, we need to consider the operators

$$\hat{U}(0) = \hat{1}_{\mathcal{H}} = \hat{1}_1 \otimes \hat{1}_2 , \quad \hat{U}_1(0) \equiv \text{Tr}_{\mathcal{H}_2} \hat{1}_{\mathcal{H}} = M_2 \hat{1}_1 ,$$

$$\hat{U}_2(0) \equiv \text{Tr}_{\mathcal{H}_1} \hat{1}_{\mathcal{H}} = M_1 \hat{1}_2 \quad \hat{U}^{\otimes}(0) = \frac{\hat{U}_1(0) \otimes \hat{U}_2(0)}{M_1 M_2} .$$

With the norms squared

$$||\hat{U}_1(0)||^2 = M_1 M_2^2 , \quad ||\hat{U}_2(0)||^2 = M_1^2 M_2 , \quad ||\hat{U}^{\otimes}(0)||^2 = M_1 M_2 ,$$

we find that $\varepsilon(0) = 0$, as it should be.

At finite time, the measure of entanglement production can become non-zero, which depends on the system Hamiltonian. In some particular cases of the latter, the evolution operator can be simplified for any finite time [29,30]. For an arbitrary Hamiltonian, one can consider the short-time behavior. Then, as $t \rightarrow 0$, to second order in t , we have

$$\hat{U}_1(t) \simeq M_2 - it \text{Tr}_{\mathcal{H}_2} H - \frac{t^2}{2} \text{Tr}_{\mathcal{H}_2} H^2 \quad \hat{U}_2(t) \simeq M_1 - it \text{Tr}_{\mathcal{H}_1} H - \frac{t^2}{2} \text{Tr}_{\mathcal{H}_1} H^2 . \quad (31)$$

Introducing the notation

$$\begin{aligned} \hat{\Delta}_1 &\equiv M_2 \text{Tr}_{\mathcal{H}_2} H^2 - (\text{Tr}_{\mathcal{H}_2} H)^2 , & \hat{\Delta}_2 &\equiv M_1 \text{Tr}_{\mathcal{H}_1} H^2 - (\text{Tr}_{\mathcal{H}_1} H)^2 , \\ \hat{\Delta}_{12} &\equiv M_1 M_2 \text{Tr}_{\mathcal{H}} H^2 - (\text{Tr}_{\mathcal{H}} H)^2 , \end{aligned} \quad (32)$$

we find

$$\begin{aligned} ||\hat{U}_1(t)||^2 &\simeq M_1 M_2^2 - (\text{Tr}_{\mathcal{H}_1} \hat{\Delta}_1) t^2 , & ||\hat{U}_2(t)||^2 &\simeq M_1^2 M_2 - (\text{Tr}_{\mathcal{H}_2} \hat{\Delta}_2) t^2 , \\ |\text{Tr}_{\mathcal{H}} \hat{U}(t)|^2 &\simeq M_1^2 M_2^2 - \Delta_{12} t^2 . \end{aligned} \quad (33)$$

Therefore

$$||\hat{U}^{\otimes}(t)||^2 \simeq M_1 M_2 - \mu t^2 , \quad (34)$$

where

$$\mu \equiv \frac{1}{M_1 M_2} \left(M_1 \text{Tr}_{\mathcal{H}_1} \hat{\Delta}_1 + M_2 \text{Tr}_{\mathcal{H}_2} \hat{\Delta}_2 - \Delta_{12} \right) .$$

Finally, we obtain the short-time behavior of the entanglement-production measure

$$\varepsilon(t) \simeq \frac{1}{2} \mu t^2 \quad (t \rightarrow 0) , \quad (35)$$

calculated to second order in t . Here, we keep in mind the natural logarithm in definition (27). Dealing with the logarithm over the base 2, we should replace μ by $\mu / \ln 2$.

At the initial stage, the entanglement production is quadratic in time.

5 Heisenberg evolutionary entanglement

As an illustration, let us consider a bipartite system characterized by spins $\mathbf{S}_j = \{S_j^\alpha\}$, with the Heisenberg interaction. Such spin ensembles represent many finite-state systems widely studied in a variety of physics applications as well as in information processing. The Hamiltonian is a sum of two terms:

$$H = H_0 + H_{int} , \quad (36)$$

where the first term has the Zeeman structure

$$H_0 = -h \left(S_1^z \otimes \hat{1}_2 + \hat{1}_1 \otimes S_2^z \right) , \quad (37)$$

and the second term describes an anisotropic Heisenberg interaction

$$H_{int} = J_1 \left(S_1^x \otimes S_2^x + S_1^y \otimes S_2^y \right) + 2J S_1^z \otimes S_2^z . \quad (38)$$

The Heisenberg model is defined for any dimensionality of spins \mathbf{S}_j . Here, we shall consider spins one-half, with the standard relation of spin components with the Pauli matrices: $S_j^\alpha = (1/2)\sigma_j^\alpha$.

Using the ladder operators $S_j^\pm \equiv S_j^x \pm S_j^y$ reduces the interaction term to the form

$$H_{int} = 2J S_1^z \otimes S_2^z + J_1 \left(S_1^+ \otimes S_2^- + S_1^- \otimes S_2^+ \right) . \quad (39)$$

The interaction parameters J and J_1 can be of any sign.

Considering the entanglement production by the evolution operator, we follow the previous sections, omitting the details of the calculational procedure that is delineated in the Appendix A. For expressions (32), we find

$$\begin{aligned} \hat{\Delta}_1 &= (h^2 + J^2 + 2J_1^2) \hat{1}_1 - 4JhS_1^z , & \hat{\Delta}_2 &= (h^2 + J^2 + 2J_1^2) \hat{1}_2 - 4JhS_2^z , \\ \Delta_{12} &= 4(2h^2 + J^2 + 2J_1^2) . \end{aligned}$$

The norm of the non-entangling evolution-operator counterpart (26), at short time, reads as

$$\|\hat{U}^\otimes(t)\|^2 \simeq 8 \frac{2 - (h^2 + J^2 + 2J_1^2)t^2}{4 - (2h^2 + J^2 + 2J_1^2)t^2} .$$

Then, for the entanglement production measure (27) at the initial stage, we obtain

$$\varepsilon(t) \simeq \frac{1}{8} (J^2 + 2J_1^2) t^2 \quad (t \rightarrow 0) , \quad (40)$$

in agreement with the quadratic in time behavior (35).

Note that at this initial stage, the evolutionary entanglement is produced by spin interactions, while an external field is not yet playing role.

6 Ising evolutionary entanglement

In order to analyze the behavior of the entanglement-production measure for all times, let us consider a system with strongly anisotropic Heisenberg interactions yielding the Ising Hamiltonian

$$\begin{aligned} H &= H_0 + H_{int} , \\ H_0 &= -h \left(S_1^z \otimes \hat{1}_2 + \hat{1}_1 \otimes S_2^z \right) , & H_{int} &= 2J S_1^z \otimes S_2^z . \end{aligned} \quad (41)$$

In view of the commutator

$$[H_0, H_{int}] = 0 ,$$

we have

$$e^{-iHt} = e^{-iH_0t} e^{-iH_{int}t} . \quad (42)$$

Expanding the exponents in Taylor series and summing back, as is explained in the Appendix B, we find for the exponent with the Zeeman term

$$e^{-iH_0t} = 1 + \frac{H_0^2}{h^2} [\cos(ht) - 1] - i \frac{H_0}{h} \sin(ht) , \quad (43)$$

while for the exponent with the interaction term, we get

$$e^{-iH_{int}t} = \cos\left(\frac{Jt}{2}\right) - 2i \frac{H_{int}}{J} \sin\left(\frac{Jt}{2}\right) . \quad (44)$$

Then the evolution operator can be represented as

$$\begin{aligned} e^{-iHt} = & \left\{ 1 + \frac{H_0^2}{h^2} [\cos(ht) - 1] \right\} \cos\left(\frac{Jt}{2}\right) - \frac{H_0}{h} \sin(ht) \sin\left(\frac{Jt}{2}\right) - \\ & - i \left\{ \frac{2H_{int}}{J} + \frac{H_0^2}{h^2} [\cos(ht) - 1] \right\} \sin\left(\frac{Jt}{2}\right) - i \frac{H_0}{h} \sin(ht) \cos\left(\frac{Jt}{2}\right) . \end{aligned} \quad (45)$$

For the partially-traced operators, defined in Eqs. (25), we have

$$\begin{aligned} \hat{U}_j(t) = & [1 + \cos(ht)] \cos\left(\frac{Jt}{2}\right) \hat{1}_j + 2S_j^z \sin(ht) \sin\left(\frac{Jt}{2}\right) + \\ & + i[1 - \cos(ht)] \sin\left(\frac{Jt}{2}\right) \hat{1}_j + 2iS_j^z \sin(ht) \cos\left(\frac{Jt}{2}\right) , \end{aligned} \quad (46)$$

where $j = 1, 2$. Their norms squared are given by the formula

$$||\hat{U}_j(t)||^2 = 4[1 + \cos(ht) \cos(Jt)] . \quad (47)$$

And for the evolution operator, we obtain the trace

$$\text{Tr}_{\mathcal{H}} \hat{U}(t) = 2[1 + \cos(ht)] \cos\left(\frac{Jt}{2}\right) + 2i[1 - \cos(ht)] \sin\left(\frac{Jt}{2}\right) , \quad (48)$$

which yields

$$|\text{Tr}_{\mathcal{H}} \hat{U}(t)|^2 = 4[1 + \cos^2(ht) + 2 \cos(ht) \cos(Jt)] . \quad (49)$$

Finally, the entanglement-production measure (27), caused by the evolution operator, is

$$\varepsilon(t) = \log \frac{\sqrt{1 + \cos^2(ht) + 2 \cos(ht) \cos(Jt)}}{1 + \cos(ht) \cos(Jt)} . \quad (50)$$

This measure, as is straightforward to check, is positive for all times $t > 0$. It tends to zero at the beginning of the evolution as

$$\varepsilon(t) \simeq \frac{J^2}{8 \ln 2} t^2 + \frac{J^2(J^2 - 12h^2)}{192 \ln 2} t^4 , \quad (51)$$

as it should be according to Eq. (35). Here we use the logarithm over the base 2. Again, we see that the first term does not depend on the field h that enters only in the higher terms.

The measure does not depend on the signs of h and J . But the existence of interactions is crucial, since without interactions there is no entanglement at all:

$$\lim_{J \rightarrow 0} \varepsilon(t) = 0. \quad (52)$$

The existence of the field h is also important. This is due to the invariance of Hamiltonian (41) with respect to the spin inversion $S_j^z \rightarrow -S_j^z$, when $h \equiv 0$, while this invariance is absent for any finite h . In the case of this invariance, when $h \equiv 0$, we have

$$\varepsilon(t) = \frac{1}{2} \log \frac{2}{1 + \cos(Jt)} \quad (h \equiv 0).$$

This expression diverges at the moments of time $\pi(1 + 2n)/J$, where $n = 0, 1, 2, \dots$. On the contrary, at these moments of time, $\varepsilon(t)$, defined by Eq. (50), is zero, when $h \neq 0$ is any finite quantity, except special points of the set of zero measure to be defined below. The singularity points correspond to the exceptional conditions when either

$$\frac{h}{J} = \frac{2p}{1 + 2n}, \quad t_1 = (1 + 2n) \frac{\pi}{J}, \quad (53)$$

or when

$$\frac{h}{J} = \frac{1 + 2n}{2p}, \quad t_2 = 2p \frac{\pi}{J}, \quad (54)$$

with $n = 0, 1, 2, \dots$ and $p = 1, 2, \dots$. For all other h , there are no singularities.

Generally, the entanglement production measure (50) is quasi-periodic, with the periods

$$T_1 = \frac{\pi}{|h|}, \quad T_2 = \frac{2\pi}{|h + J|}, \quad T_3 = \frac{2\pi}{|h - J|}, \quad (55)$$

except when the periods are commensurable. Thus, when h/J is an irreducible rational number $h/J = p/q$, where p and q both are odd numbers, then expression (50) is periodic, with the period $T = \pi q$. And when h/J is rational, such that $h/J = p/q$, where one of the integers is even, while the other is odd, then function (50) is periodic, with the period $T = 2\pi q$.

The typical temporal behavior of measure (50), as a function of time measured in units of $1/J$, is shown in Figs. 1 and 2. The logarithm is taken with respect to base 2. The field h is measured in units of J . Figure 1 shows the cases of periodic behavior, while Fig. 2 illustrates quasi-periodic entanglement-production.

By changing the system parameters, it is possible to regulate the evolutionary process of entanglement production.

7 Conclusion

When a system is in a disentangled state but one needs to transfer it into an entangled state, two ways are possible, which can be classified as external and internal. One way is when entanglement is generated in a system by resorting to externally imposed appropriate

transformations. Some of the related cases have been considered earlier. For example, by an external alternating field it is possible to generate multiple entangled modes in a Bose-Einstein condensate. The other possibility is to allow for the system to naturally evolve according to the evolution law prescribed by the evolution operator. It is this second way that is studied in the present paper. The entanglement production generated by the evolution operator has not been considered in previous literature.

Entanglement production, generated by an evolution operator $\hat{U}(t)$, and quantified by the entanglement production measure $\varepsilon(\hat{U}(t))$, is investigated. As illustrations, we consider the bipartite systems with spin interactions of the Heisenberg and Ising types. Such spin objects are typical for many finite-level or finite-state physical systems that can be employed for information processing. The measure of entanglement production oscillates in time, being in general quasi-periodic. The existence of interactions is crucial for this measure to be nonzero. Without interactions, no entanglement is produced.

The evolutional entanglement, produced by the evolution operator, as studied in the present paper, is different from the entanglement produced by a time-dependent statistical operator $\hat{\rho}(t)$ of a nonequilibrium system, as considered in Refs. [10–12, 17]. The entanglement production measure, analyzed in the latter papers, has been

$$\varepsilon(\hat{\rho}(t)) = \log \frac{\|\hat{\rho}(t)\|}{\|\hat{\rho}^{\otimes}(t)\|} ,$$

where, according to the general definition (8), the disentangled, or distilled, statistical operator is

$$\hat{\rho}^{\otimes}(t) = \bigotimes_{i=1}^N \hat{\rho}_i(t) ,$$

with the partial operators

$$\hat{\rho}_i(t) = \text{Tr}_{\mathcal{H}/\mathcal{H}_i} \hat{\rho}(t) .$$

Also, the entanglement production, generated by quantum operations, should not be confused with the state entanglement that is quantified by other measures [1–5]. For example, the entanglement of a state, corresponding to a statistical operator $\hat{\rho}(t)$, can be quantified by the relative entropy

$$D(t) = \text{Tr}_{\mathcal{H}} \hat{\rho}(t) \ln \frac{\hat{\rho}(t)}{\hat{\rho}^{\otimes}(t)} ,$$

that is also called the Kullback-Leibler distance, since it shows the distance of the state $\hat{\rho}(t)$ from the distilled state $\hat{\rho}^{\otimes}(t)$. When several distillations are admissible, one considers the minimum of the above distance.

By studying the entanglement production, caused by the evolution operator, it is possible to evaluate the period of time during which the considered system would evolve from an initial disentangled state to an entangled state. This method of following the natural evolution of the system provides an alternative to the procedure of creating entanglement by means of external transformations.

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Appendix A. Evolutional entanglement production for Heisenberg interactions

Calculating the operator norms, we meet the powers of the Hamiltonian, which are represented in the standard symmetrized form. For instance, the squares of the sums of two operators are given by the expressions

$$\left(\hat{A}_i + \hat{B}_i\right)^2 = \hat{A}_i^2 + \hat{B}_i^2 + \hat{A}_i\hat{B}_i + \hat{B}_i\hat{A}_i ,$$

$$\left(\hat{A}_1 \otimes \hat{A}_2 + \hat{B}_1 \otimes \hat{B}_2\right)^2 = \hat{A}_1^2 \otimes \hat{A}_2^2 + \hat{B}_1^2 \otimes \hat{B}_2^2 + \hat{A}_1\hat{B}_1 \otimes \hat{A}_2\hat{B}_2 + \hat{B}_1\hat{A}_1 \otimes \hat{B}_2\hat{A}_2 .$$

In that way, for Hamiltonians (37) and (38), we have

$$H_0^2 = \frac{h^2}{2} \left(\hat{1}_{\mathcal{H}} + 4S_1^z \otimes S_2^z \right) ,$$

$$H_{int}^2 = \frac{1}{4} \left(J^2 + 2J_1 \right) \hat{1}_{\mathcal{H}} - 2J_1^2 S_1^z \otimes S_2^z - JJ_1 \left(S_1^+ \otimes S_2^- + S_1^- \otimes S_2^+ \right) ,$$

$$H_0 H_{int} = H_{int} H_0 = \frac{1}{2} J H_0 .$$

Then for Hamiltonian (36), we get

$$H^2 = H_0^2 + J H_0 + H_{int}^2 .$$

Under spins one-half, the basis can be taken as a set of two vectors, corresponding to spin up and spin down. So that $M_1 = M_2 = 2$.

The following traces are found:

$$\begin{aligned} \text{Tr}_{\mathcal{H}_1} H_0 &= -2hS_2^z , & \text{Tr}_{\mathcal{H}_2} H_0 &= -2hS_1^z , & \text{Tr}_{\mathcal{H}_1} H_{int} &= \text{Tr}_{\mathcal{H}_2} H_{int} = 0 , \\ \text{Tr}_{\mathcal{H}_1} H_0^2 &= h^2 \hat{1}_2 , & \text{Tr}_{\mathcal{H}_2} H_0^2 &= h^2 \hat{1}_1 , \\ \text{Tr}_{\mathcal{H}_1} H_{int}^2 &= \left(\frac{1}{2} J^2 + J_1^2 \right) \hat{1}_2 , & \text{Tr}_{\mathcal{H}_2} H_{int}^2 &= \left(\frac{1}{2} J^2 + J_1^2 \right) \hat{1}_1 , \\ \text{Tr}_{\mathcal{H}_1} H &= -2hS_2^z , & \text{Tr}_{\mathcal{H}_2} H &= -2hS_1^z , \\ \text{Tr}_{\mathcal{H}_1} H^2 &= \left(h^2 + \frac{1}{2} J^2 + J_1^2 \right) \hat{1}_2 - 2JhS_2^z , & \text{Tr}_{\mathcal{H}_2} H^2 &= \left(h^2 + \frac{1}{2} J^2 + J_1^2 \right) \hat{1}_1 - 2JhS_1^z , \\ \text{Tr}_{\mathcal{H}} H &= 0 , & \text{Tr}_{\mathcal{H}} H^2 &= 2h^2 + J^2 + 2J_1^2 . \end{aligned}$$

This results in measure (40).

Appendix B. Evolutional entanglement production for Ising interactions

Expanding the exponent $\exp(-iH_0t)$, we meet the terms

$$H_0^3 = h^2 H_0, \quad H_0^4 = h^2 H_0^2, \quad H_0^5 = h^4 H_0, \quad H_0^6 = h^4 H_0^2,$$

and so on, resulting in the relations

$$H_0^{2n} = h^{2(n-1)} H_0^2, \quad H_0^{2n+1} = h^{2n} H_0,$$

which lead to Eq. (43).

Expanding the exponent $\exp(-iH_{int}t)$, we find

$$H_{int}^2 = \left(\frac{J}{2}\right)^2, \quad H_{int}^3 = \left(\frac{J}{2}\right)^2 H_{int}, \quad H_{int}^4 = \left(\frac{J}{2}\right)^4, \quad H_{int}^5 = \left(\frac{J}{2}\right)^4 H_{int},$$

and so on, which gives the equations

$$H_{int}^{2n} = \left(\frac{J}{2}\right)^{2n}, \quad H_{int}^{2n+1} = \left(\frac{J}{2}\right)^{2n} H_{int}.$$

These relations yield Eq. (44).

References

- [1] C.P. Williams and S.H. Clearwater, *Explorations in Quantum Computing* (Springer, New York, 1998).
- [2] M.A. Nielsen and I.L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, New York, 2000).
- [3] V. Vedral, Rev. Mod. Phys. **74**, 197 (2002).
- [4] M. Keyl, Phys. Rep. **369**, 431 (2002).
- [5] M. Wilde, *Quantum Information Theory* (Cambridge University, Cambridge, 2013).
- [6] V.I. Yukalov and D. Sornette, Laser Phys. **23**, 105502 (2013).
- [7] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301 (2000).
- [8] P. Zanardi, Phys. Rev. A **63**, 040304 (2001).
- [9] V.I. Yukalov, Phys. Rev. Lett. **90**, 167905 (2003).
- [10] V.I. Yukalov, Phys. Rev. A **68**, 022109 (2003).
- [11] V.I. Yukalov, Laser Phys. **14**, 1403 (2004).
- [12] V.I. Yukalov and E.P. Yukalova, Phys. Rev. A **73**, 022335 (2006).
- [13] V. Vedral, J. Phys. Conf. Ser. **143**, 012010 (2009).
- [14] E. Martin-Martinez, E.C. Brown, W. Donnelly, and A. Kempf, Phys. Rev. A **88**, 052310 (2013).
- [15] A. Ström, H. Johannesson, and P. Recher, Phys. Rev. B **91**, 245406 (2015).
- [16] W. Chen, D.N. Shi, and D.Y. Xing, Sci. Rep. **5**, 7607 (2015).
- [17] V.I. Yukalov, Mod. Phys. Lett. B **17**, 95 (2003).
- [18] V.I. Yukalov and E.P. Yukalova, Laser Phys. **16**, 354 (2006).
- [19] V.I. Yukalov and D. Sornette, Phys. At. Nucl. **73**, 559 (2010).
- [20] F. Mintert, A.R. Carvalho, M. Kus, and A. Buchleitner, Phys. Rep. **415**, 207 (2005).
- [21] E.J. Heller, Phys. Rev. A **35**, 1360 (1987).
- [22] D.E. Logan and P.G. Wolynes, J. Chem. Phys. **93**, 4994 (1990).
- [23] D.M. Basko, I.L. Aleiner, and B.L. Altshuler, Ann. Phys. (N.Y.) **321**, 1126 (2006).
- [24] A. Pal and D.A. Huse, Phys. Rev. B **82**, 174411 (2010).

- [25] D.A. Huse, R. Nandkishore, V. Oganesyan, A. Pal, and S.L. Sondhir, Phys. Rev. B **88**, 014206 (2013).
- [26] J.L. Birman, R.G. Nazmitdinov, and V.I. Yukalov, Phys. Rep. **526**, 1 (2013).
- [27] J. Weidmann, *Linear Operators in Hilbert Spaces* (Springer, New York, 1980).
- [28] R. Bhatia, *Matrix Analysis* (Springer, Berlin, 1997).
- [29] D.S. Bernstein and W. So, IEEE Trans. Autom. Control **38**, 1228 (1993).
- [30] V. Ramakrishna and H. Zhou, J. Phys. A, **39**, 3021 (2006).

Figure Captions

Figure 1. The entanglement-production measure, for the case of periodic evolution, as a function of time measured in units of $1/J$, for different fields: (a) $h/J = 1$ (the period is π); (b) $h/J = 5/7$ (the period is 7π); (c) $h/J = 7$ (the period is π); (d) $h/J = 8$ (the period is 2π).

Figure 2. The measure of evolutionary entanglement production, illustrating quasi-periodic behavior, for different fields: (a) $h/J = \sqrt{2}$; (b) $h/J = \sqrt{3}/2$; (c) $h/J = \sqrt{5}$; (d) $h/J = \sqrt{7}$.

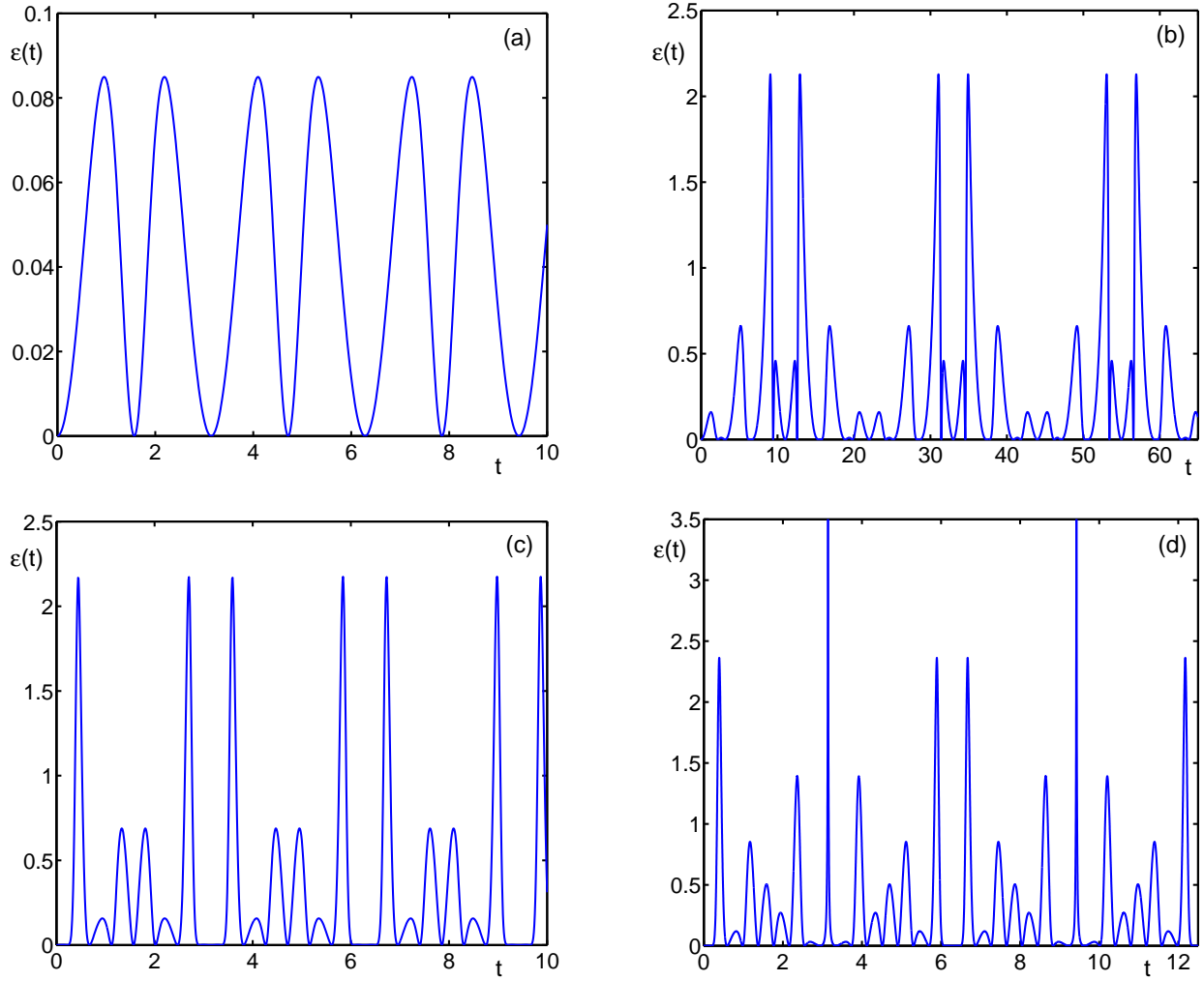


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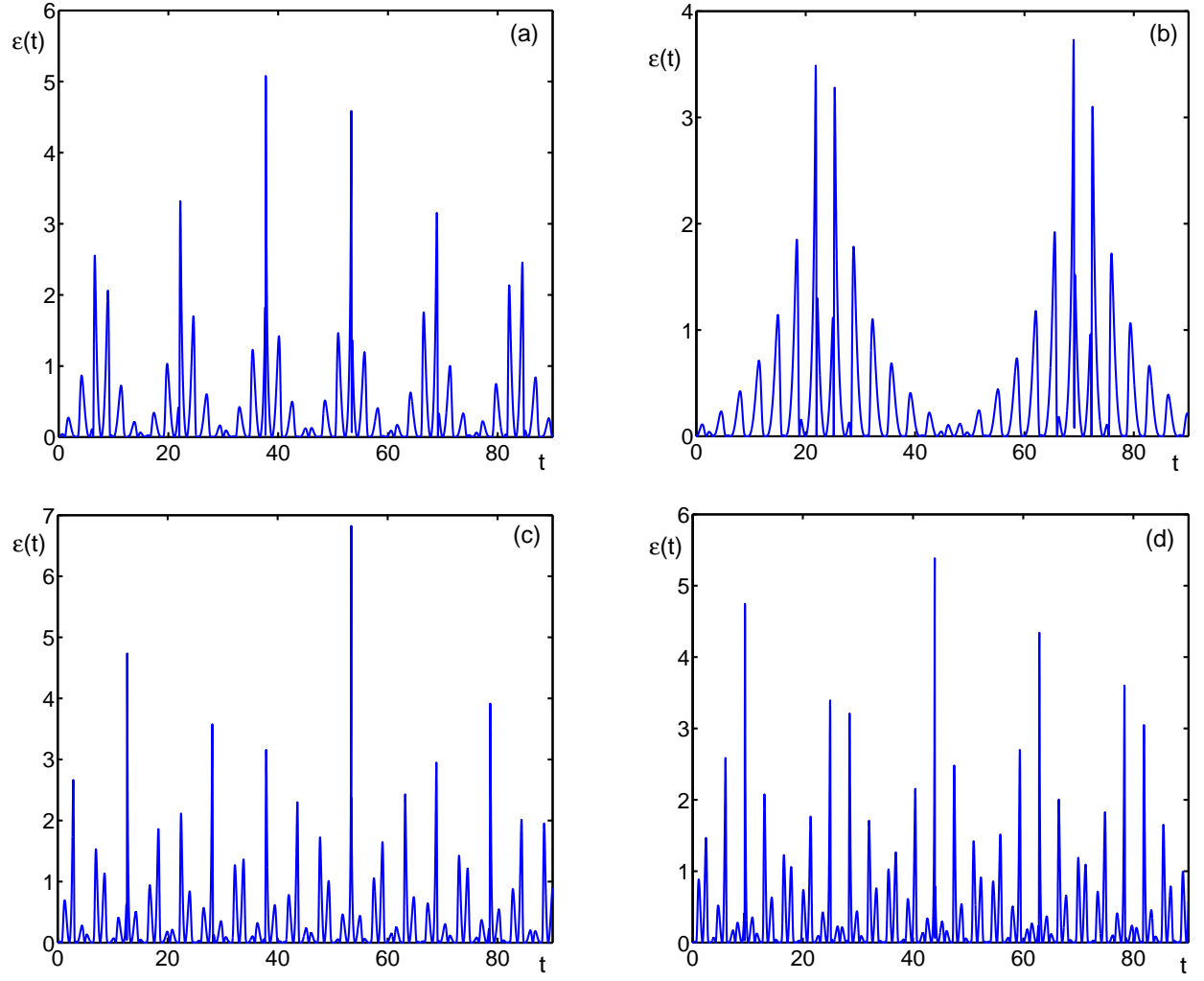


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